Calc BC: Gold #8

① (E) Think about graphs of each

② $\dfrac{8x-7}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} \Rightarrow \dfrac{3}{x-2} + \dfrac{5}{x+1}$ (B)

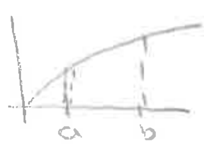
$x = 2 \Rightarrow A = \frac{9}{3} = 3$

$x = -1 \Rightarrow B = \frac{-15}{-3} = 5$

③ $\int_0^\infty e^{-2t}\,dt = \lim_{b\to\infty} \int_0^b e^{-2t}\,dt$

$= \lim_{b\to\infty} -\frac{1}{2}e^{-2t}\Big]_0^b = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2b} + \frac{1}{2}\right]$

$= \lim_{b\to\infty}\left[-\frac{1}{2e^{2b}} + \frac{1}{2}\right] = \frac{1}{2}$ (C)

④ 

$L = \int_a^b \sqrt{1 + \left[\frac{d}{dx}(\sqrt{x})\right]^2}\,dx$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$ $\quad = \int_a^b \sqrt{1 + \frac{1}{4x}}\,dx$

(E)

(5) $\frac{dy}{dx} = y \cdot \sec^2 x$

$\frac{1}{y}\, dy = \sec^2 x \, dx$

$\ln|y| = \tan x + c$

$y = Ce^{\tan x}$

$5 = C$

$y = 5e^{\tan x}$ (C)

(6) $r(t) = \langle \sin^{-1} t, (t+4)^2 \rangle$

$v(t) = \langle \frac{1}{\sqrt{1-t^2}}, 2(t+4) \rangle$

$v(.6) = \langle \frac{1}{\sqrt{.64}}, 2(4.6) \rangle$

$= \langle \frac{1}{.8}, 9.2 \rangle$ (B)

(7) $L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^4 \sqrt{(3t^2)^2 + (2e^{2t})^2}\, dt$

(C)

(8) Geometric, $r = -\frac{2}{3}$

$S_n = \frac{a_n}{1-r} \Rightarrow \frac{\frac{9}{8}}{1 + \frac{2}{3}}$

$= \frac{9}{8} \cdot \frac{3}{5} = \frac{27}{40}$

(E)

(9) MVT for integrals

$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$

$(b-a) \cdot f(c) = \int_a^b f(x)\, dx$

(C)

(10) $\int_1^2 (t-1)^{-1/3}\, dt$

$= \frac{3}{2}(t-1)^{2/3}\Big]_1^2$

$= \frac{3}{2}(1-0) = \frac{3}{2}$ (B)

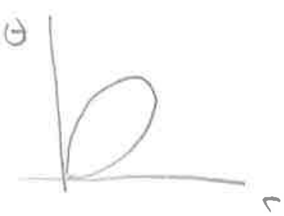
(11) $\frac{dy}{dx} = \frac{3t^2}{2t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2}{2t}\right)}{2t} = \frac{\frac{2t\cdot 6t - 2(3t^2)}{4t^2}}{2t}$$

$$= \frac{6t^2}{8t^3} = \frac{3}{4t}$$ (A)

(12) (A)

(13)



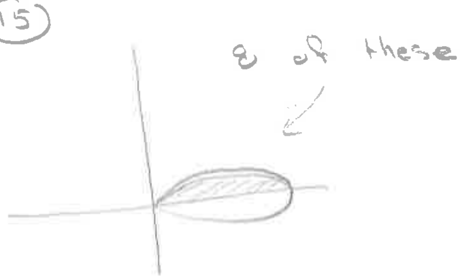
$A = \int_0^{\pi/2} \frac{1}{2}\left[\sin(2\theta)\right]^2 d\theta = \frac{\pi}{8}$ (D)

P.S. I started the w/ calc problems early

(14) $A = \frac{1}{2}\int_0^{\pi/3} \left[2\sin(3\theta)\right]^2 d\theta = 2\int_0^{\pi/3} \sin^2(3\theta)\, d\theta$ (C)

(Note: $\pi/3$ is where $2\sin(3\theta)$ 1st equals zero)

(15)



$A = 16\cdot\int_0^{\pi/8} \frac{1}{2}(5\cos(4\theta))^2\, d\theta$

$= 200\int_0^{\pi/8} \cos^2(4\theta)\, d\theta$

$= 39.2699 \approx 12.5\pi$ (D)

(16) $\vec{v}(t) = \langle -\cos t + c_1, \ -e^{-t} + c_2 \rangle$

$\vec{v}(0) = 0 \Rightarrow -\cos(0) + c_1 = 0 \qquad -e^0 + c_2 = 0$

$c_1 = 1 \qquad c_2 = 1$

$\vec{v}(t) = \langle -\cos t + 1, \ -e^{-t} + 1 \rangle$

$|\vec{v}(t)| = \sqrt{(-\cos t + 1)^2 + (-e^{-t})^2}$ graph to find max on $0 \le t \le 5$ 2.217

(C)

(17) Diverges (E). But why? Here's why...

$$\int_2^4 \frac{1}{(x-3)^2}\,dx = \int_2^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx$$

$$\int_2^3 \frac{1}{(x-3)^2}\,dx = \lim_{a \to 3^-} \int_2^a \frac{1}{(x-3)^2}\,dx$$

$$= \lim_{a \to 3^-} -\left(\frac{1}{x-3}\right)\Big]_2^a$$

$$= \lim_{a \to 3^-} \left[-\frac{1}{a-3} - 1\right]$$

$= \infty$, diverges!

(18) $a(t) = -5.28$

$v(t) = -5.28\,t + C \;:\; v(0) = 56$

$56 = C$

$v(t) = -5.28\,t + 56$

$v(4.5) = 32.24$ (C)

(19) $\frac{dP}{dh} = kP$

$\frac{1}{P}\,dP = k\,dh$

$\ln|P| = kh + C_1$

$|P| = e^{kh + C}$

$P = Ce^{kh} \;:\; P = 30$ when $h = 0$

$C = 30$

$P = 30e^{kh}$ $\quad P = 24$ when $h = 1000$

$24 = 30e^{1000k}$

$k = \frac{\ln \frac{24}{30}}{1000} = -3.39 \times 10^{-5}$

$P(5000) = 30e^{5000k}$

$= 25.32$ in (B)

(20) a) You graph

b) use symmetry, intersect at $\Theta = \frac{\pi}{3}$

$$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3\cos\Theta)^2 - (1+\cos\Theta)^2\right] d\Theta$$

$$= \int_0^{\pi/3} \left(9\cos^2\Theta - 1 - 2\cos\Theta - \cos^2\Theta\right) d\Theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\Theta - 2\cos\Theta - 1\right) d\Theta$$

$$= \int_0^{\pi/3} \left[4(1+\cos 2\Theta) - 2\cos\Theta - 1\right] d\Theta$$

$$= \left[4\left(\Theta + \tfrac{1}{2}\sin(2\Theta)\right) - 2\sin\Theta - \Theta\right]_0^{\pi/3}$$

$$= \left(4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \sqrt{3} - \frac{\pi}{3}\right) - (0)$$

$$= \frac{4\pi}{3} + \sqrt{3} - \sqrt{3} - \frac{\pi}{3}$$

$$= \pi$$

(21) See answer sheet

(22) $x = 2t^3 - 3t^2, \quad y = t^3 - 12t$

a) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 12}{6t^2 - 6t} = \frac{t^2 - 4}{2t^2 - 2t}$

b) $x(-1) = -5, \quad y(-1) = 11$

$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{-3}{4}$

$y - 11 = -\frac{3}{4}(x + 5)$

c) horizontal tangent $\Rightarrow \frac{dy}{dx} = 0$

$t = \pm 2$

$t = -2$; $x(-2) = -28$
$y(-2) = 16$
$(-28, 16)$

$t = 2$; $x(2) = 4$
$y(2) = -16$
$(4, -16)$

vertical tangent $\Rightarrow \frac{dy}{dx}$ is undefined

$2t^2 - 2t = 0$
$2t(t-1) = 0$
$t = 0, t = 1$

$t = 0$: $(0, 0)$

$t = 1$: $(-1, -11)$

(23) $r = 3(1-\cos\theta)$

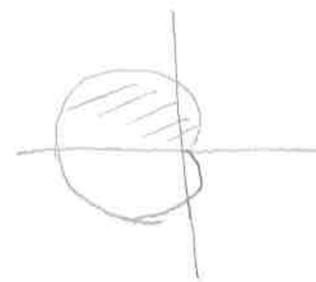
$x = 3\cos\theta(1-\cos\theta) = 3\cos\theta - 3\cos^2\theta$

$y = 3\sin\theta(1-\cos\theta) = 3\sin\theta - 3\sin\theta\cdot\cos\theta$

a)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta - 3\cos^2\theta + 3\sin^2\theta}{-3\sin\theta + 6\cos\theta\cdot\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{3\pi}{2}} = \frac{0-0+3}{3+0} = 1$$



b) $$A = 2\cdot\frac{1}{2}\int_0^{\pi} (3(1-\cos\theta))^2\, d\theta$$

$$= 9\int_0^{\pi} (1-2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 9\int_0^{\pi} (1-2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta))\, d\theta$$

$$= 9\left[\theta - 2\sin\theta + \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)\right]_0^{\pi}$$

$$= 9\left[(\pi - 0 + \tfrac{1}{2}(\pi+0)) - (0 - 0 + \tfrac{1}{2}(0+0))\right]$$

$$= 9\left(\frac{3\pi}{2}\right) = \frac{27\pi}{2}$$

(23) c)

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = 2\int_0^{\pi} \sqrt{(3(1-\cos\theta))^2 + (3\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{9(1-2\cos\theta+\cos^2(\theta)) + 9\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{9 - 18\cos\theta + 9\cos^2\theta + 9\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{9(1-2\cos\theta) + 9(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 6\int_0^{\pi} \sqrt{2-2\cos\theta}\, d\theta$$

$$= 6\int_0^{\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 6\int_0^{\pi} \sqrt{\frac{4(1-\cos\theta)}{2}}\, d\theta$$

$$\sin\tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$$

$$= 12\int_0^{\pi} \sqrt{\frac{1-\cos\theta}{2}}\, d\theta$$

$$= 12\int_0^{\pi} \sin\tfrac{1}{2}\theta\, d\theta$$

$$= 12\left[-2\cos\tfrac{1}{2}\theta\right]_0^{\pi}$$

$$= 12(0+2) = \underline{\underline{24}}$$